The SIR Model Family

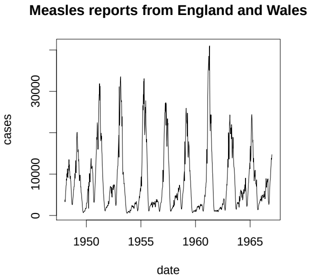
Jonathan Dushoff, McMaster University

`http://lalashan.mcmaster.ca/DushoffLab`

MMED 2016

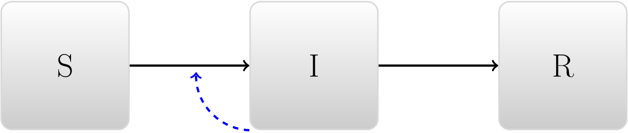
`http://www.ici3d.org/mmed/`

Dynamic modeling
Connects scales



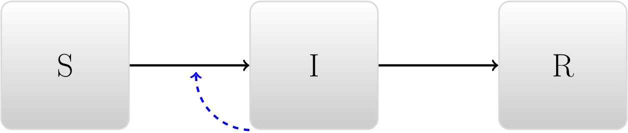
Compartmental models

Divide people into categories:



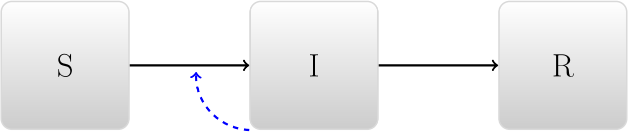
- Susceptible $\rightarrow$ Infectious $\rightarrow$ Recovered

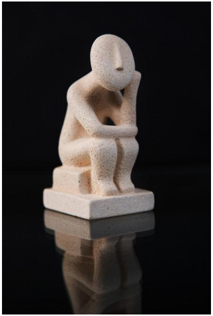
What determines transition rates?



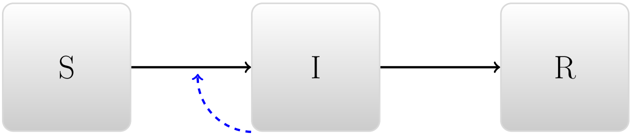
- People get better independently
- People get infected by infectious people

Conceptual modeling



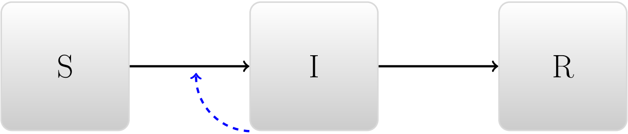


Conceptual modeling



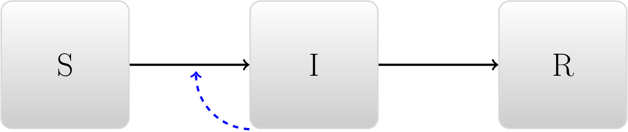
- What is the final result?
- When does disease increase, decrease?

Dynamic implementation



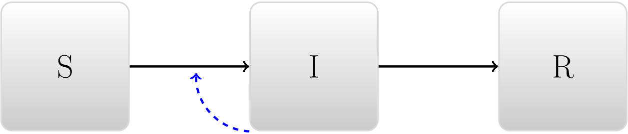
- Requires assumptions about recovery and transmission
- The *conceptually simplest* implementation uses Ordinary Differential Equations (ODEs)
  - Other options may be more realistic
  - Or simpler in practice

Recovery



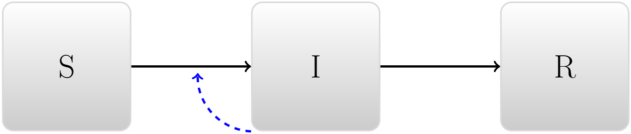
- Infectious people recover at *per capita* rate γ
  - Total recovery rate is γI
  - Mean time infectious is $D = 1/\gamma$

Transmission



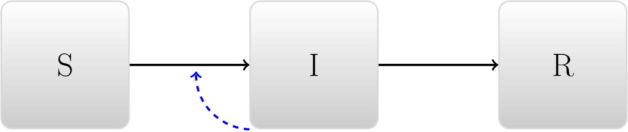
- Susceptible people get infected by:
  - Going around and contacting people (rate c)
  - Some of these people are infectious (proportion I/N)
  - Some of these contacts are effective (proportion p)
- Per capita rate of becoming infected is $cpI/N \equiv \beta I/N$
- Population-level transmission rate is $\mathcal{T} = \beta SI/N$

Another perspective on transmission



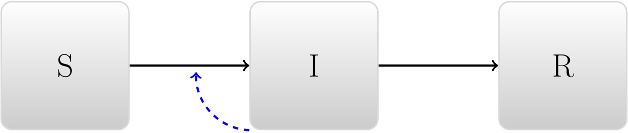
- Infectious people infect others by:
  - Going around and contacting people (rate c)
  - Some of these people are susceptible (proportion S/N)
  - Some of these contacts are effective (proportion p)
- Per capita rate of infecting others is $cpS/N \equiv \beta S/N$
- Population-level transmission rate is $\mathcal{T} = \beta SI/N$

ODE implementation



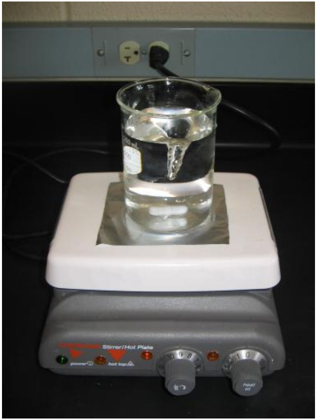
$$\begin{aligned}
\frac{dS}{dt} &= -\beta\frac{SI}{N}\\
\frac{dI}{dt} &= \beta\frac{SI}{N} - \gamma I\\
\frac{dR}{dt} &= \gamma I
\end{aligned}$$

Spreadsheet implementation



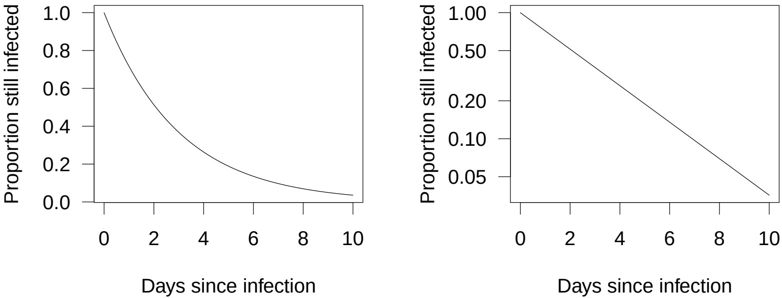
http://tinyurl.com/SIR-MMED-2016

ODE assumptions



- Lots and lots of people
- Perfectly mixed

ODE assumptions



- Waiting times are exponentially distributed
- Rarely realistic

Scripts vs. spreadsheets

	Susceptibles people	Infectious people	Removed people	Total people
0	999	1	0	
1	998.6004	1.1996	0.2	
2	998.121231584064	1.438848415936	0.43992	
3	997.546773522873	1.72553679393953	0.7276896831872	
4	996.858252058318	2.06895089970737	1.07279704197511	
5	996.033271747327	2.48014103075661	1.48658722191658	
6	995.045150553223	2.97223401870901	1.9826154280679	
7	993.862147734573	3.56079003361749	2.5770622318097	
8	992.446573962396	4.26420579907117	3.2892202385332	
9	990.753775388012	5.10416321364044	4.14206139834744	
10	988.730987798368	6.1061181605567	5.16289404107552	
11	986.316064502168	7.29981782464567	6.38411767318686	
12	983.436093466813	8.71982529507146	7.844081238116	
13	980.005937097253	10.4060166056164	9.58804629713029	

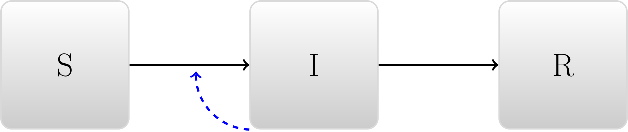
Sheet1 Sheet2

```
return(as.list(
    Sdot = - beta*S*I/N,
    Idot = S*I/N - I/D,
    Rdot = I/D
))
~
~
~
~
~
~
```

More about transmission

- $\beta = pc$
  - What is a contact?
  - What is the probability of transmission?
- Sometimes this decomposition is clear
- But usually it's not

Population sizes



$$
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\frac{dS}{dt} &= -\beta \frac{SI}{N} \\
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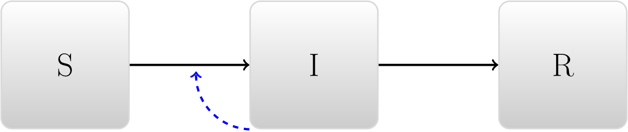
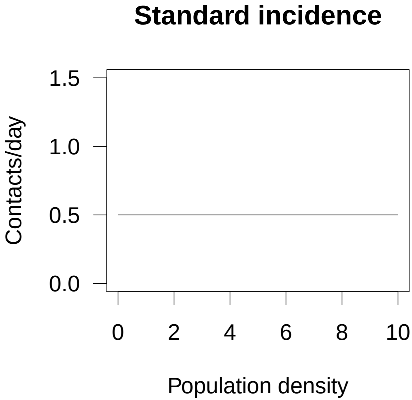
\frac{dR}{dt} &= \gamma I
\end{aligned}
$$

Population sizes



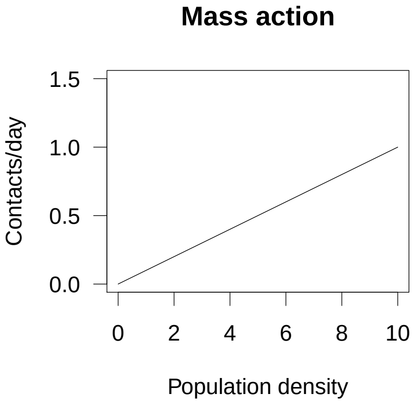
$$
\begin{aligned}
\frac{dS}{dt} &= -\beta(N)\frac{SI}{N} \\
\frac{dI}{dt} &= \beta(N)\frac{SI}{N} - \gamma I \\
\frac{dR}{dt} &= \gamma I
\end{aligned}
$$

Standard incidence



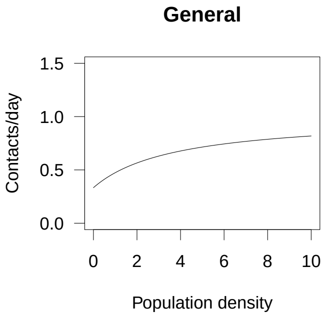
- $\beta(N) = \beta_0$
- $\mathcal{T} = \frac{\beta_0 SI}{N}$
- Also known as *frequency-dependent* transmission

Mass action



- $\beta(N) = \beta_1 N$
- $\mathcal{T} = \beta_1 S I$
- Also known as *density-dependent* transmission

Other

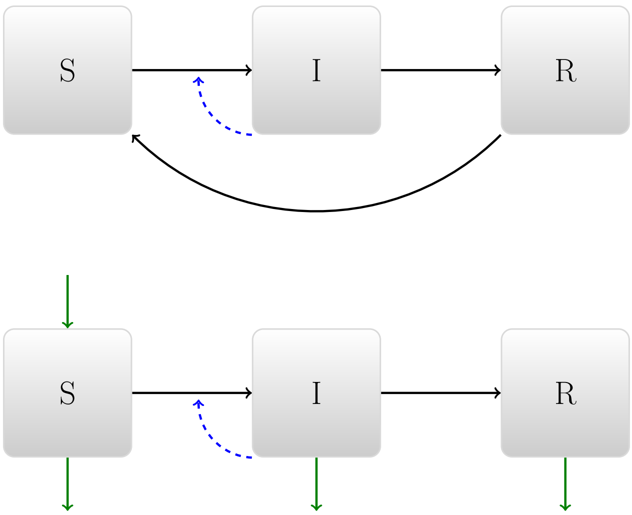


- May not go to zero when N does
- May not go to ∞ when N does

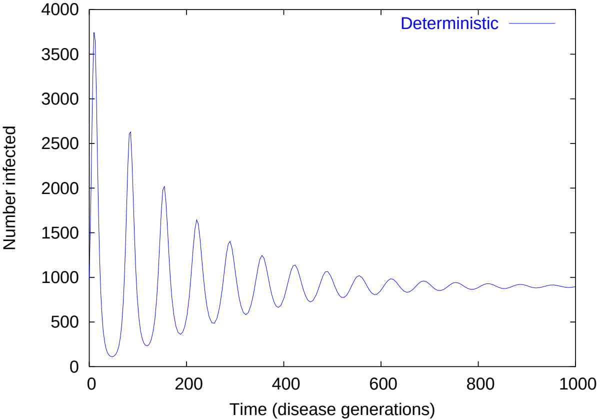
Digression – units

- $\mathcal{T} = \beta SI/N$: [ppl/time]
- β : [1/time]
  - $\beta/\gamma = \beta D$: [1]
  - Standard incidence, β_0 : [1/time]
  - Mass-action incidence, β_1 : [1/(people $\cdot$ time)]

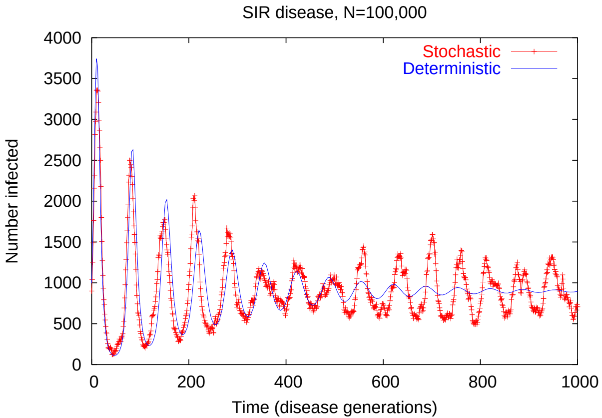
Closing the circle



Tendency to oscillate



With individuality



Summary

- Dynamics are an essential tool to link scales
- Very simple models can provide useful insights
- More complex models can provide more detail, but also require more assumptions, and more choices

Conclusions from simple models

- There is a link between individual-level processes and population-level outcomes
- The reproductive number (number of cases per case) is a key quantity
  - Disease increases when $\mathcal{R} > 1$
  - Decreases when $\mathcal{R} < 1$
- Oscillation
  - If susceptibles are replenished, diseases have a tendency to oscillate
  - These oscillations tend to be damped (get smaller through time)
- These conclusions from **simple** models help *guide* our understanding of more complicated models